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Cover Page Footnote

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Bayesian and Semi-Bayesian Estimation of the Parameters of Generalized Inverse Weibull Distribution

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Bayesian and semi-Bayesian estimators of parameters of the generalized inverse Weibull distribution are obtained using Jeffreys' prior and informative prior under specific assumptions of loss function. Using simulation, the relative efficiency of the proposed estimators is obtained under different set-ups. A real life example is also given.

Keywords: Generalized inverse Weibull distribution, Bayesian, loss functions, priors

Introduction

The three-parameter Generalized Inverse Weibull distribution (GIWD), introduced by de Gusmão, Ortega, and Cordeiro (2011), is a positively skewed distribution used to model the income data and, because of its ability of possessing decreasing and unimodal failure rate, is also useful in reliability and biological studies. GIWD is the generalization of various well-known and useful distributions. Most of the sub-cases of GIWD are families of inverse distributions, which play an important role in many applications (Drapella, 1993; Jiang, Murthy, & Ji, 2001; Nelson, 1982; Khan, Pasha, & Pasha, 2008; Zaharim, Najid, Razali, & Sopian, 2009) and are also fitted to income-related data. These distributions have two parameters but, in order to fit better at the tails (Lubrano, 2014; Kakwani, 1980), a distribution with three parameters (GIWD) is used in the present study.

The cdf of GIWD is

$$F(x) = e^{-\gamma \left(\frac{\alpha}{x}\right)^\beta}, \quad x > 0; \alpha, \beta, \gamma > 0 \quad (1)$$

where α is scale parameter and β, γ are shape parameters.

The pdf of GIWD is

$$f(x) = \gamma \beta \alpha^\beta x^{-(\beta+1)} e^{-\gamma \left(\frac{\alpha}{x}\right)^\beta} \quad (2)$$

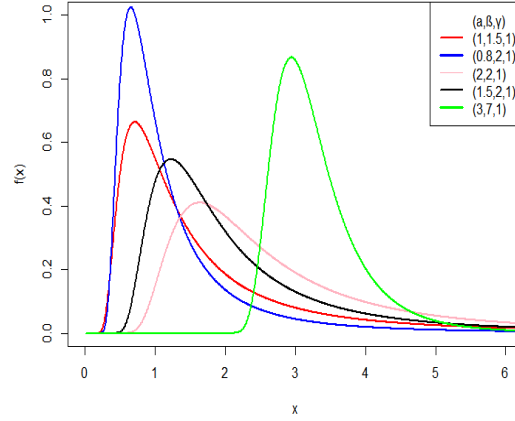


Figure 1. Plot of pdf of GIWD for various combinations of parameters

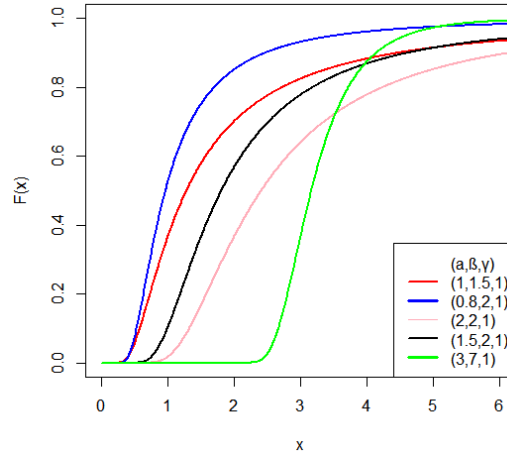


Figure 2. Plot of cdf of GIWD for various combinations of parameters

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Sub-models:

- For $\alpha = 1$, it reduces to inverse Weibull (IW) distribution.
- For $\gamma = \alpha = 1$, it reduces to Fréchet (F) distribution.
- For $\beta = 2$, $\alpha = 1$, it reduces to inverse Rayleigh (IR) distribution.
- For $\gamma = \beta = 1$, it reduces to inverse exponential (IE) distribution.

Properties of Generalized Inverse Weibull Distribution

The Mean and variance of GIWD have been obtained by de Gusmão et al. (2011) and are given by

$$\gamma^{\frac{1}{\beta}} \alpha \Gamma(1 - \beta^{-1}) \text{ and } \gamma^{\frac{2}{\beta}} \alpha^2 \left[\Gamma(1 - 2\beta^{-1}) - \left(\Gamma(1 - \beta^{-1}) \right)^2 \right]$$

respectively.

De Gusmão et al. (2011) have also obtained the estimators of the parameters of GIWD using a maximum likelihood method of estimation (non-Bayesian approach); they are obtained by using the likelihood function given by

$$L(x | \gamma, \alpha, \beta) = \gamma^n \beta^n \alpha^{n\beta} \left(\prod_{i=1}^n x_i^{-(\beta+1)} \right) \exp \left(-\gamma \alpha^\beta \sum_{i=1}^n x_i^{-\beta} \right) \quad (3)$$

The MLEs of α and γ are

$$\hat{\alpha} = \left(\frac{n}{\gamma \sum_{i=1}^n x_i^{-\beta}} \right)^{\frac{1}{\beta}}$$

$$\hat{\gamma} = \frac{n}{\alpha^\beta \sum_{i=1}^n x_i^{-\beta}}$$

To obtain the MLE of β equation (4) is solved using iterative numerical techniques (Newton–Raphson algorithm):

$$\frac{n}{\beta} + n \ln \alpha - \sum_{i=1}^n \ln x_i - \gamma \alpha^\beta \sum_{i=1}^n x_i^{-\beta} \left(\ln \alpha - \ln \left(\sum_{i=1}^n x_i \right) \right) = 0 \quad (4)$$

In the classical setup, MLEs are used to estimate the parameters of GIWD. However, in the Bayesian context, estimators of these parameters still await attention of the researchers. In the Bayesian setup, one may use a full-Bayesian approach or semi-Bayesian approach, referred to as generalized maximum likelihood estimators. In the present paper, both these approaches are used to estimate the parameters of GIWD.

For Bayesian estimation, the choice of priors and loss functions are two important aspects. Both informative and non-informative priors are used for the study in case of semi-Bayesian and full-Bayesian approach. In full-Bayesian approach both symmetrical and asymmetrical loss functions are used for the study; however no loss function is required for estimating the parameters in case of semi-Bayesian approach.

Prior and Posterior Densities

Informative Prior for the Parameters of Generalized Inverse Weibull Distribution

The informative prior depends on the elicitation of a prior distribution based on pre-existing scientific knowledge in the area of investigation. This information may be available from previous investigations or from non-statistician experts.

Assuming independence among parameters α , β , γ of GIWD, priors for α , β , γ are chosen to be gamma distributions as follows:

$$\begin{aligned}\gamma &\sim \text{Gamma}(a_1, b_1) \\ \alpha &\sim \text{Gamma}(a_2, b_2) \\ \beta &\sim \text{Gamma}(a_3, b_3)\end{aligned}$$

where $\text{Gamma}(a_i, b_i)$ denotes a gamma distribution given by

$$f(y; a_i, b_i) = \frac{b_i^{a_i} y^{a_i-1} \exp(-yb_i)}{\Gamma(a_i)}$$

where $y > 0$, $a_i > 0$, $b_i > 0$, $i = 1, 2, 3$, and a_i and b_i are the hyperparameters.

Prior density functions for α , β , and γ are

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$$g(\alpha; a_2, b_2) \propto \alpha^{a_2-1} \exp(-\alpha b_2) \quad (5)$$

$$g(\beta; a_3, b_3) \propto \beta^{a_3-1} \exp(-\beta b_3) \quad (6)$$

$$g(\gamma; a_1, b_1) \propto \gamma^{a_1-1} \exp(-\gamma b_1) \quad (7)$$

The joint prior distribution for α , β , and γ is

$$g_I(\alpha, \beta, \gamma) \propto \gamma^{a_1-1} \alpha^{a_2-1} \beta^{a_3-1} \exp(-\gamma b_1 - \alpha b_2 - \beta b_3) \quad (8)$$

The joint posterior distribution for the parameters α , β , γ for GIWD is given by

$$\begin{aligned} g_I^*(\alpha, \beta, \gamma | x) &= \frac{L(\gamma, \alpha, \beta) g(\gamma; a_1, b_1) g(\alpha; a_2, b_2) g(\beta; a_3, b_3)}{\int \int \int_0^\infty L(\gamma, \alpha, \beta) g(\gamma; a_1, b_1) g(\alpha; a_2, b_2) g(\beta; a_3, b_3) \partial \alpha \partial \beta \partial \gamma} \\ &\propto \gamma^{n+a_1-1} \beta^{n+a_3-1} \alpha^{n\beta+a_2-1} \prod_{i=1}^n x_i^{-(\beta+1)} \exp\left(\left(-\gamma \alpha^\beta \sum_{i=1}^n x_i^{-\beta}\right) - \gamma b_1 - \alpha b_2 - \beta b_3\right) \end{aligned} \quad (9)$$

Non-Informative Prior (Jeffreys' Prior) for the Parameters of Generalized Inverse Weibull Distribution

Jeffreys' (1946) prior based on Fisher's information is defined as

$$\pi(\theta) \propto \sqrt{I(\theta)}$$

where $I(\theta) = -E\left[\frac{\partial^2}{\partial \theta^2} \ln L(\theta | x)\right]$ is Fisher's information based on likelihood function $L(\theta | x)$.

In the case of GIWD, Jeffreys' prior is the square root of Fisher's information matrix of order 3×3 , given by

$$I(\theta) = -E \begin{bmatrix} \frac{\partial^2 (\ln L(x))}{\partial \alpha^2} & \frac{\partial^2 (\ln L(x))}{\partial \alpha \partial \beta} & \frac{\partial^2 (\ln L(x))}{\partial \alpha \partial \gamma} \\ \frac{\partial^2 (\ln L(x))}{\partial \beta \partial \alpha} & \frac{\partial^2 (\ln L(x))}{\partial \beta^2} & \frac{\partial^2 (\ln L(x))}{\partial \beta \partial \gamma} \\ \frac{\partial^2 (\ln L(x))}{\partial \gamma \partial \alpha} & \frac{\partial^2 (\ln L(x))}{\partial \gamma \partial \beta} & \frac{\partial^2 (\ln L(x))}{\partial \gamma^2} \end{bmatrix}$$

where

$$\begin{aligned} \frac{\partial^2 (\ln L(x))}{\partial \alpha^2} &= -\frac{n\beta}{\alpha^2} - \beta(\beta-1)\gamma\alpha^{\beta-2} \sum_{i=1}^n x_i^{-\beta} \\ \frac{\partial^2 (\ln L(x))}{\partial \beta^2} &= -\frac{n}{\beta^2} - \gamma \left(\ln \alpha - \ln \left(\sum_{i=1}^n x_i \right) \right)^2 \alpha^\beta \sum_{i=1}^n x_i^{-\beta} \\ \frac{\partial^2 (\ln L(x))}{\partial \gamma^2} &= -\frac{n}{\gamma^2} \\ \frac{\partial^2 (\ln L(x))}{\partial \alpha \partial \beta} &= \frac{n}{\alpha} - \gamma \alpha^{\beta-1} \sum_{i=1}^n x_i^{-\beta} \left[\beta \left(\ln \alpha - \ln \left(\sum_{i=1}^n x_i \right) \right) + 1 \right] = \frac{\partial^2 (\ln L(x))}{\partial \beta \partial \alpha} \\ \frac{\partial^2 (\ln L(x))}{\partial \alpha \partial \gamma} &= -\beta \alpha^{\beta-1} \sum_{i=1}^n x_i^{-\beta} = \frac{\partial^2 (\ln L(x))}{\partial \gamma \partial \alpha} \\ \frac{\partial^2 (\ln L(x))}{\partial \beta \partial \gamma} &= -\alpha^\beta \sum_{i=1}^n x_i^{-\beta} \left(\ln \alpha - \ln \left(\sum_{i=1}^n x_i \right) \right) = \frac{\partial^2 (\ln L(x))}{\partial \gamma \partial \beta} \end{aligned}$$

The expected value of double derivatives is not in a closed form, hence the explicit expression for the Jeffreys' prior is not obtained. For simplicity it is assumed that all the three parameters are independent; therefore joint prior in case of Jeffreys' prior (Guure, Ibrahim, & Ahmed, 2012; Singh, Singh, & Kumar, 2011) is written as

$$g_J(\alpha, \beta, \gamma) \propto \frac{1}{\gamma \alpha \beta} \quad (10)$$

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Note that $g_J(\alpha, \beta, \gamma)$ is a special case of the informative prior by taking the hyperparameters to be zero ($a_1 = b_1 = a_2 = b_2 = a_3 = b_3 = 0$).

The joint posterior distribution for α, β, γ in the case of GIWD is given by

$$g_J^*(\alpha, \beta, \gamma | x) \propto \gamma^{n-1} \beta^{n-1} \alpha^{n\beta-1} \prod_{i=1}^n x_i^{-(\beta+1)} \exp\left(-\gamma \alpha^\beta \sum_{i=1}^n x_i^{-\beta}\right) \quad (11)$$

Full-Bayesian Approach

To obtain the Bayesian estimators of α, β , and γ in the case of GIWD, it may be noted that the joint posterior distributions of α, β , and γ , for both informative and non-informative priors, is a ratio form that involves an integration in the denominator and cannot be reduced to a closed form. Hence the evaluation of the posterior expectation for obtaining the Bayesian estimators of α, β, γ will be tedious. Among the various methods suggested to approximate the ratio of integrals of the above form, the simplest one is Lindley's approximation method (Lindley, 1980), which approaches the ratio of the integrals as a whole and produces a single numerical result.

$$I(x) = E[u(\theta_1, \theta_2, \theta_3)] = \frac{\int_{(\theta_1, \theta_2, \theta_3)} u(\theta_1, \theta_2, \theta_3) e^{L(\theta_1, \theta_2, \theta_3) + G(\theta_1, \theta_2, \theta_3)} d(\theta_1, \theta_2, \theta_3)}{\int_{(\theta_1, \theta_2, \theta_3)} e^{L(\theta_1, \theta_2, \theta_3) + G(\theta_1, \theta_2, \theta_3)} d(\theta_1, \theta_2, \theta_3)}$$

where $L(\theta_1, \theta_2, \theta_3)$ is the log of likelihood and $G(\theta_1, \theta_2, \theta_3)$ is the log of the joint prior of θ_1, θ_2 , and θ_3 .

$I(x)$ can also be written as

$$\begin{aligned} I(x) &= E[u(\theta_1, \theta_2, \theta_3)] \\ &= u(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3) + (u_1 a_1 + u_2 a_2 + u_3 a_3 + a_4 + a_5) + \frac{1}{2} \left[A(u_1 \sigma_{11} + u_2 \sigma_{12} + u_3 \sigma_{13}) + B(u_1 \sigma_{21} + u_2 \sigma_{22} + u_3 \sigma_{23}) + C(u_1 \sigma_{31} + u_2 \sigma_{32} + u_3 \sigma_{33}) \right] \end{aligned} \quad (12)$$

where $\hat{\theta}_1, \hat{\theta}_2$, and $\hat{\theta}_3$ are the MLEs of θ_1, θ_2 , and θ_3 , respectively,

$$a_i = \rho_1 \sigma_{i1} + \rho_2 \sigma_{i2} + \rho_3 \sigma_{i3}, \quad i = 1, 2, 3 \quad (13)$$

$$a_4 = u_{12}\sigma_{12} + u_{13}\sigma_{13} + u_{23}\sigma_{23} \quad (14)$$

$$a_5 = \frac{1}{2}(u_{11}\sigma_{11} + u_{22}\sigma_{22} + u_{33}\sigma_{33}) \quad (15)$$

$$A = \sigma_{11}L_{111} + 2\sigma_{12}L_{121} + 2\sigma_{13}L_{131} + 2\sigma_{23}L_{231} + \sigma_{22}L_{221} + \sigma_{33}L_{331} \quad (16)$$

$$B = \sigma_{11}L_{112} + 2\sigma_{12}L_{122} + 2\sigma_{13}L_{132} + 2\sigma_{23}L_{232} + \sigma_{22}L_{222} + \sigma_{33}L_{332} \quad (17)$$

$$C = \sigma_{11}L_{113} + 2\sigma_{12}L_{123} + 2\sigma_{13}L_{133} + 2\sigma_{23}L_{233} + \sigma_{22}L_{223} + \sigma_{33}L_{333} \quad (18)$$

subscripts 1, 2, 3, on the right-hand sides above refer to θ_1 , θ_2 , and θ_3 , respectively, ρ is the logarithm of the joint prior density,

$$\rho_i = \frac{\partial \rho}{\partial \theta_i}, \quad i = 1, 2, 3$$

$$u_i = \frac{\partial u(\theta_1, \theta_2, \theta_3)}{\partial \theta_i}, \quad i = 1, 2, 3$$

$$u_{ij} = \frac{\partial^2 u(\theta_1, \theta_2, \theta_3)}{\partial \theta_i \partial \theta_j}, \quad i, j = 1, 2, 3$$

$$L_{ij} = \frac{\partial^2 L(\theta_1, \theta_2, \theta_3)}{\partial \theta_i \partial \theta_j}, \quad i, j = 1, 2, 3$$

$$L_{ijk} = \frac{\partial^3 L(\theta_1, \theta_2, \theta_3)}{\partial \theta_i \partial \theta_j \partial \theta_k}, \quad i, j, k = 1, 2, 3$$

and σ_{ij} is the $(i, j)^{\text{th}}$ element of the inverse of the matrix having elements $\{-L_{ij}\}$.

Remark: The expression $I(x)$ leads to the approximate expression for the Bayesian estimators.

Lindley's Approximation for Generalized Inverse Weibull Distribution in case of Informative Prior

The various terms as listed above are derived below for the three parameters α , β , and γ in the case of GIWD using informative prior.

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The log of the joint prior density (ρ) in the case of informative prior (8) is

$$\begin{aligned}\rho &= \log(g(\alpha, \beta, \gamma)) \\ &= (a_1 - 1) \log \gamma + (a_2 - 1) \log \alpha + (a_3 - 1) \log \beta - b_1 \gamma - b_2 \alpha - b_3 \beta\end{aligned}$$

$$\rho_1 = \frac{\partial \rho}{\partial \alpha} = \frac{a_2 - 1}{\alpha} - b_2 \quad (19)$$

$$\rho_2 = \frac{\partial \rho}{\partial \beta} = \frac{a_3 - 1}{\beta} - b_3 \quad (20)$$

$$\rho_3 = \frac{\partial \rho}{\partial \gamma} = \frac{a_1 - 1}{\gamma} - b_1 \quad (21)$$

The log-likelihood, using equation (3), is given by

$$\begin{aligned}L(\alpha, \beta, \gamma) &= \log(L(x|\gamma, \alpha, \beta)) \\ &= n \log \gamma + n \log \beta + n \beta \log \alpha - (\beta + 1) \sum_{i=1}^n \log x_i - \gamma \alpha^\beta \sum_{i=1}^n x_i^{-\beta}\end{aligned}$$

$$L_{11} = \frac{\partial^2 L(\alpha, \beta, \gamma)}{\partial \alpha^2} = \frac{-n\beta}{\alpha^2} - \beta(\beta - 1) \gamma \alpha^{\beta-2} \sum_{i=1}^n x_i^{-\beta} \quad (22)$$

$$L_{22} = \frac{\partial^2 L(\alpha, \beta, \gamma)}{\partial \beta^2} = -\frac{n}{\beta^2} - \gamma \left(\log \alpha - \log \sum_{i=1}^n x_i \right)^2 \alpha^\beta \sum_{i=1}^n x_i^{-\beta} \quad (23)$$

$$L_{33} = \frac{\partial^2 L(\alpha, \beta, \gamma)}{\partial \gamma^2} = \frac{-n}{\gamma^2} \quad (24)$$

$$L_{12} = \frac{\partial^2 L(\alpha, \beta, \gamma)}{\partial \alpha \partial \beta} = \frac{n}{\alpha} - \alpha^{\beta-1} \sum_{i=1}^n x_i^{-\beta} \left[\beta \left(\log \alpha - \log \sum_{i=1}^n x_i \right) + 1 \right] = L_{21} \quad (25)$$

$$L_{13} = \frac{\partial^2 L(\alpha, \beta, \gamma)}{\partial \alpha \partial \gamma} = -\beta \alpha^{\beta-1} \sum_{i=1}^n x_i^{-\beta} = L_{31} \quad (26)$$

$$L_{23} = \frac{\partial^2 L(\alpha, \beta, \gamma)}{\partial \beta \partial \gamma} = -\alpha^\beta \sum_{i=1}^n x_i^{-\beta} \left(\log \alpha - \log \sum_{i=1}^n x_i \right) = L_{32} \quad (27)$$

$$L_{111} = \frac{\partial^3 L(\alpha, \beta, \gamma)}{\partial \alpha^3} = \frac{2n\beta}{\alpha^3} - \gamma\beta(\beta-1)(\beta-2)\alpha^{\beta-3} \sum_{i=1}^n x_i^{-\beta} \quad (28)$$

$$L_{222} = \frac{\partial^3 L(\alpha, \beta, \gamma)}{\partial \beta^3} = -\frac{2n}{\beta^3} - \gamma \left(\log \alpha - \log \sum_{i=1}^n x_i \right)^3 \alpha^\beta \sum_{i=1}^n x_i^{-\beta} \quad (29)$$

$$L_{333} = \frac{\partial^3 L(\alpha, \beta, \gamma)}{\partial \gamma^3} = \frac{2n}{\gamma^3} \quad (30)$$

$$L_{113} = \frac{\partial^3 L(\alpha, \beta, \gamma)}{\partial \alpha^2 \partial \gamma} = -\beta(\beta-1)\alpha^{\beta-2} \sum_{i=1}^n x_i^{-\beta} = L_{131} = L_{311} \quad (31)$$

$$\begin{aligned} L_{112} &= \frac{\partial^3 L(\alpha, \beta, \gamma)}{\partial \alpha^2 \partial \beta} \\ &= \frac{-n}{\alpha^2} - \gamma \alpha^{\beta-2} \sum_{i=1}^n x_i^{-\beta} \left[(2\beta-1) + \beta(\beta-1) \left(\log \alpha - \log \sum_{i=1}^n x_i \right) \right] \\ &= L_{121} = L_{211} \end{aligned} \quad (32)$$

$$\begin{aligned} L_{122} &= \frac{\partial^3 L(\alpha, \beta, \gamma)}{\partial \alpha \partial \beta^2} \\ &= -\gamma \alpha^{\beta-1} \sum_{i=1}^n x_i^{-\beta} \left(\log \alpha - \log \sum_{i=1}^n x_i \right) \left(2 + \beta \left(\log \alpha - \log \sum_{i=1}^n x_i \right) \right)^2 \\ &= L_{221} = L_{212} \end{aligned} \quad (33)$$

$$\begin{aligned} L_{123} &= \frac{\partial^3 L(\alpha, \beta, \gamma)}{\partial \alpha \partial \beta \partial \gamma} \\ &= -\alpha^{\beta-1} \sum_{i=1}^n x_i^{-\beta} \left(1 + \beta \left(\log \alpha - \log \sum_{i=1}^n x_i \right) \right) = L_{321} = L_{213} \end{aligned} \quad (34)$$

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$$L_{133} = \frac{\partial^3 L(\alpha, \beta, \gamma)}{\partial \alpha \partial \gamma^2} = 0 = L_{313} = L_{331} \quad (35)$$

$$\begin{aligned} L_{223} &= \frac{\partial^3 L(\alpha, \beta, \gamma)}{\partial \beta^2 \partial \gamma} \\ &= -\left(\log \alpha - \log \sum_{i=1}^n x_i\right)^2 \alpha^\beta \sum_{i=1}^n x_i^{-\beta} = L_{322} = L_{323} \end{aligned} \quad (36)$$

$$L_{233} = \frac{\partial^3 L(\alpha, \beta, \gamma)}{\partial \beta \partial \gamma^2} = 0 = L_{332} = L_{323} \quad (37)$$

The expression $a_i = \rho_1 \sigma_{i1} + \rho_2 \sigma_{i2} + \rho_3 \sigma_{i3}$, $i = 1, 2, 3$, hence in this case

$$a_1 = \rho_1 \sigma_{11} + \rho_2 \sigma_{12} + \rho_3 \sigma_{13} \quad (38)$$

$$a_2 = \rho_1 \sigma_{21} + \rho_2 \sigma_{22} + \rho_3 \sigma_{23} \quad (39)$$

$$a_3 = \rho_1 \sigma_{31} + \rho_2 \sigma_{32} + \rho_3 \sigma_{33} \quad (40)$$

Using these general expressions of Lindley's approximation, Bayesian estimators for α, β, γ for different loss functions are derived below.

Bayesian Estimators of α, β, γ for Informative Prior using Lindley's Approximation in Case of Squared Error Loss Function (SELF)

The squared error loss function is

$$L(\theta) = (\theta - \hat{\theta})^2$$

where $\hat{\theta}$ is the estimator of θ . The Bayes estimator of θ under the SELF is

$$\hat{\theta}_{\text{SELF}} = E(\theta)$$

where expectation is taken with respect to posterior density. Note the Bayesian estimator of θ using Lindley's approximation is directly given by the expression $I(x)$ in equation (12).

Bayesian Estimator of α under Squared Error Loss Function

In this case, $u(\theta_1, \theta_2, \theta_3) = u(\alpha, \beta, \gamma)$ is function of α only and it is given by $u_{as} = u(\alpha, \beta, \gamma) = \alpha$, since parameters α, β, γ are assumed to be independent. Hence, using equation (12),

$$I(x) = E[u(\alpha, \beta, \gamma)] = E(\alpha)$$

the first derivative of u_{as} with respect to α is

$$u_{as1} = \frac{\partial u_{as}}{\partial \alpha} = 1$$

all other $u_{ij} = 0$ for $i, j = 1, 2, 3$ because it is a function of α only. The expressions for the $a_i, i = 1, 2, 3$ are given in equations (38)-(40), and

$$\begin{aligned} a_4 &= (u_{12}\sigma_{12} + u_{13}\sigma_{13} + u_{23}\sigma_{23}) = 0 \\ a_5 &= \frac{1}{2}(u_{11}\sigma_{11} + u_{22}\sigma_{22} + u_{33}\sigma_{33}) = 0 \end{aligned}$$

Using equation (12), the expression for $I(x)$ is derived as

$$I(x) = \hat{\alpha} + (u_{as1}a_1) + \frac{1}{2}[A\sigma_{11} + B\sigma_{21} + C\sigma_{31}]$$

where

$$\begin{aligned} A &= \sigma_{11}L_{111} + 2\sigma_{12}L_{121} + 2\sigma_{13}L_{131} + 2\sigma_{23}L_{231} + \sigma_{22}L_{221} + \sigma_{33}L_{331} \\ B &= \sigma_{11}L_{112} + 2\sigma_{12}L_{122} + 2\sigma_{13}L_{132} + 2\sigma_{23}L_{232} + \sigma_{22}L_{222} + \sigma_{33}L_{332} \\ C &= \sigma_{11}L_{113} + 2\sigma_{12}L_{123} + 2\sigma_{13}L_{133} + 2\sigma_{23}L_{233} + \sigma_{22}L_{223} + \sigma_{33}L_{333} \end{aligned}$$

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The detailed expression for the various L_{ijk} for $i, j, k = 1, 2, 3$ are obtained in the above equations (28)-(37). Moreover, σ_{ij} is the $(i, j)^{\text{th}}$ element of the inverse of the matrix of L_{ij} , i.e.

$$\sigma_{ij} = \frac{-1}{L_{ij}}, \quad i, j = 1, 2$$

where the L_{ij} for $i, j = 1, 2$ are computed in equations (22)-(27) above. Therefore, the Bayesian estimator of α under the squared error loss function is

$$\begin{aligned} \hat{\alpha}_{\text{SELF}} &= \hat{\alpha} + \left(\frac{a_2 - 1}{\hat{\alpha}} - b_2 \right) \sigma_{11} + \left(\frac{a_3 - 1}{\hat{\beta}} - b_3 \right) \sigma_{12} + \left(\frac{a_1 - 1}{\hat{\gamma}} - b_1 \right) \sigma_{13} \\ &\quad + \frac{1}{2} [A\sigma_{11} + B\sigma_{21} + C\sigma_{31}] \end{aligned}$$

In the similar way, Bayesian estimators of β ($\hat{\beta}_{\text{SELF}}$) and γ ($\hat{\gamma}_{\text{SELF}}$) under the squared error loss function are given as

$$\begin{aligned} \hat{\beta}_{\text{SELF}} &= \hat{\beta} + \left(\frac{a_2 - 1}{\hat{\alpha}} - b_2 \right) \sigma_{21} + \left(\frac{a_3 - 1}{\hat{\beta}} - b_3 \right) \sigma_{22} + \left(\frac{a_1 - 1}{\hat{\gamma}} - b_1 \right) \sigma_{23} \\ &\quad + \frac{1}{2} [A\sigma_{12} + B\sigma_{22} + C\sigma_{32}] \\ \hat{\gamma}_{\text{SELF}} &= \hat{\gamma} + \left(\frac{a_2 - 1}{\hat{\alpha}} - b_2 \right) \sigma_{31} + \left(\frac{a_3 - 1}{\hat{\beta}} - b_3 \right) \sigma_{32} + \left(\frac{a_1 - 1}{\hat{\gamma}} - b_1 \right) \sigma_{33} \\ &\quad + \frac{1}{2} [A\sigma_{13} + B\sigma_{23} + C\sigma_{33}] \end{aligned}$$

The squared error loss functions for α, β, γ are

$$\begin{aligned} L(\alpha) &= (\alpha - \hat{\alpha}_{\text{SELF}})^2 \\ L(\beta) &= (\beta - \hat{\beta}_{\text{SELF}})^2 \\ L(\gamma) &= (\gamma - \hat{\gamma}_{\text{SELF}})^2 \end{aligned}$$

Bayesian Estimators of β and γ for Informative Prior using Lindley's Approximation in Case of Linear Exponential Loss Function (LINEX)

The LINEX loss function is appropriate in case of shape parameter. So, it is not appropriate for α , but for β and γ we make use of the LINEX loss function and obtain Bayesian estimators of β and γ under this set-up.

The LINEX loss function is given by

$$L(\hat{\theta}, \theta) = e^{c(\hat{\theta} - \theta)} - c(\hat{\theta} - \theta) - 1, \quad c \neq 0$$

and the Bayes estimator of θ under the LINEX loss function is

$$\hat{\theta}_{\text{LINEX}} = -\frac{1}{c} \ln \left[E(e^{-c\theta}) \right], \quad c \neq 0$$

Using Lindley's approximation, the Bayesian estimator of θ under the LINEX loss function is given by

$$\hat{\theta}_{\text{LINEX}} = -\frac{1}{c} \ln [I(x)]$$

where $I(x)$ is the expression given by equation (12).

Bayesian Estimator of β under LINEX Loss Function

The Bayesian estimator of β under the LINEX loss function is given by

$$\hat{\beta}_{\text{LINEX}} = -\frac{1}{c} \ln \left[E(e^{-c\beta}) \right]$$

Therefore

$$I(x) = E[u(\alpha, \beta, \gamma)] = E(e^{-c\beta})$$

In this case, $u(\alpha, \beta, \gamma)$ is function of β only and is defined as

$$u_{\beta l} = u(\alpha, \beta, \gamma) = e^{-c\beta}$$

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with the first and second derivatives of $u_{\beta l}$ with respect to β given by

$$u_{\beta l 2} = \frac{\partial u_{\beta l}}{\partial \beta} = -ce^{-c\beta}$$

$$u_{\beta l 22} = \frac{\partial^2 u_{\beta l}}{\partial \beta^2} = c^2 e^{-c\beta}$$

As $u(\alpha, \beta, \gamma)$ is function of β ,

$$a_4 = u_{12}\sigma_{12} + u_{13}\sigma_{13} + u_{23}\sigma_{23} = 0$$

and

$$a_5 = \frac{1}{2}(u_{\beta l 22}\sigma_{22}) = \frac{c^2 e^{-c\beta}}{2}\sigma_{22}$$

$$\begin{aligned} I(x) &= u(\hat{\alpha}, \hat{\beta}, \hat{\gamma}) + u_{\beta l 2}a_2 + a_5 + \frac{u_{\beta l 2}}{2}[A\sigma_{12} + B\sigma_{22}C\sigma_{32}] \\ &= e^{-c\hat{\beta}} - ce^{-c\hat{\beta}} \left(\left(\frac{a_2 - 1}{\hat{\alpha}} - b_2 \right) \sigma_{21} + \left(\frac{a_3 - 1}{\hat{\beta}} - b_3 \right) \sigma_{22} + \left(\frac{a_3 - 1}{\hat{\gamma}} - b_1 \right) \sigma_{23} \right) \\ &\quad + \frac{c^2 e^{-c\hat{\beta}}}{2}\sigma_{22} - \frac{ce^{-c\hat{\beta}}}{2}[A\sigma_{12} + B\sigma_{22}C\sigma_{32}] \\ &= e^{-c\hat{\beta}} \left\{ 1 - c \left(\left(\frac{a_2 - 1}{\hat{\alpha}} - b_2 \right) \sigma_{21} + \left(\frac{a_3 - 1}{\hat{\beta}} - b_3 \right) \sigma_{22} + \left(\frac{a_3 - 1}{\hat{\gamma}} - b_1 \right) \sigma_{23} \right) \right. \\ &\quad \left. + \frac{c^2}{2}\sigma_{22} - \frac{c}{2}[A\sigma_{12} + B\sigma_{22}C\sigma_{32}] \right\} \end{aligned}$$

Hence, the Bayesian estimator of β is given by

$$\hat{\beta}_{\text{LINEX}} = -\frac{1}{c} \ln [E(e^{-c\beta})]$$

Putting the value of $I(x)$ in the above equation,

$$\hat{\beta}_{\text{LINEX}} = \hat{\beta} - \frac{1}{c} \ln \left\{ 1 - c \left[\left(\frac{a_2 - 1}{\hat{\alpha}} - b_2 \right) \sigma_{21} + \left(\frac{a_3 - 1}{\hat{\beta}} - b_3 \right) \sigma_{22} + \left(\frac{a_3 - 1}{\hat{\gamma}} - b_1 \right) \sigma_{23} \right. \right. \\ \left. \left. - \frac{c^2}{2} \sigma_{22} + \frac{c}{2} [A\sigma_{12} + B\sigma_{22}C\sigma_{32}] \right] \right\}$$

The terms in the estimator are the same as explained earlier.

Bayesian Estimator of γ under LINEX Loss Function

$$\hat{\gamma}_{\text{LINEX}} = \hat{\gamma} - \frac{1}{c} \ln \left\{ 1 - c \left[\left(\frac{a_2 - 1}{\hat{\alpha}} - b_2 \right) \sigma_{31} + \left(\frac{a_3 - 1}{\hat{\beta}} - b_3 \right) \sigma_{32} + \left(\frac{a_3 - 1}{\hat{\gamma}} - b_3 \right) \sigma_{23} \right. \right. \\ \left. \left. - \frac{c^2}{2} \sigma_{33} + \frac{c}{2} [A\sigma_{13} + B\sigma_{23}C\sigma_{33}] \right] \right\}$$

The LINEX loss functions for β and γ are

$$L(\hat{\beta}_{\text{LINEX}}, \beta) = e^{c(\hat{\beta}_{\text{LINEX}} - \beta)} - c(\hat{\beta}_{\text{LINEX}} - \beta) - 1, \quad c \neq 0 \\ L(\hat{\gamma}_{\text{LINEX}}, \gamma) = e^{c(\hat{\gamma}_{\text{LINEX}} - \gamma)} - c(\hat{\gamma}_{\text{LINEX}} - \gamma) - 1, \quad c \neq 0$$

Bayesian Estimators of α for Informative Prior using Lindley's Approximation in Case of General Entropy Loss Function (GELF)

The LINEX loss function is not as appropriate for estimation of scale parameter as it is for shape parameter (Basu & Ebrahimi, 1991; Parsian, Sanjari, & Nematollahi, 1993). So, for estimation of α , an entropy loss function is used.

The general entropy loss function is defined as

$$L(\theta, \hat{\theta}) \propto \left(\frac{\hat{\theta}}{\theta} \right)^{b_e} - b_e \ln \left(\frac{\hat{\theta}}{\theta} \right) - 1$$

The Bayes estimator of θ under general entropy loss is given as

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$$\hat{\theta}_{\text{GELF}} = \left[E_{\theta} \left(\theta^{-b_e} \right) \right]_{b_e}^{-1}$$

Using Lindley's approximation method, the Bayesian estimator of θ under the general entropy loss function is

$$\hat{\theta}_{\text{GELF}} = \left[I(x) \right]_{b_e}^{-1}$$

Bayesian Estimator of α under General Entropy Loss Function

The Bayesian estimator of α under the general entropy loss function is given by

$$\hat{\alpha}_{\text{GELF}} = \left[E \left(\alpha^{-b_e} \right) \right]_{b_e}^{-1}$$

$$I(x) = E \left[u(\alpha, \beta, \gamma) \right] = E \left(\alpha^{-b_e} \right)$$

is function of α only, and $u(\alpha, \beta, \gamma)$ is denoted by $u_{\alpha e}$, i.e.

$$u_{\alpha e} = u(\alpha, \beta, \gamma) = \alpha^{-b_e}$$

The first and second derivatives of $u_{\alpha e}$ with respect to α are given by

$$u_{\alpha e1} = \frac{\partial u_{\alpha e}}{\partial \alpha} = -b_e \alpha^{-b_e-1}$$

$$u_{\alpha e11} = \frac{\partial^2 u_{\alpha e}}{\partial \alpha^2} = b_e (b_e + 1) \alpha^{-b_e-2}$$

$$a_5 = \frac{1}{2} (u_{\alpha e11} \sigma_{11}) = \frac{b_e (b_e + 1) \alpha^{-b_e-2}}{2} \sigma_{11}$$

$$I(x) = E \left(\alpha^{-b_e} \right)$$

$$= u(\hat{\alpha}, \hat{\beta}, \hat{\gamma}) + u_{\alpha e1} a_1 + a_5 + \frac{1}{2} [A u_{\alpha e1} \sigma_{11} + B u_{\alpha e1} \sigma_{21} + C u_{\alpha e1} \sigma_{31}]$$

$$\begin{aligned}
&= \hat{\alpha}^{-b_e} - b_e \hat{\alpha}^{-b_e-1} \left(\left(\frac{a_2-1}{\hat{\alpha}} - b_2 \right) \sigma_{11} + \left(\frac{a_3-1}{\hat{\beta}} - b_3 \right) \sigma_{12} + \left(\frac{a_1-1}{\hat{\gamma}} - b_1 \right) \sigma_{13} \right) \\
&\quad + \frac{b_e(b_e+1)\hat{\alpha}^{-b_e-2}}{2} \sigma_{11} - \frac{b_e \hat{\alpha}^{-b_e-1}}{2} [A\sigma_{11} + B\sigma_{21} + C\sigma_{31}] \\
&= \left(\hat{\alpha}^{-b_e} \left\{ 1 - \frac{b_e}{\hat{\alpha}} \left[\left(\frac{a_2-1}{\hat{\alpha}} - b_2 \right) \sigma_{11} + \left(\frac{a_3-1}{\hat{\beta}} - b_3 \right) \sigma_{12} + \left(\frac{a_1-1}{\hat{\gamma}} - b_1 \right) \sigma_{13} \right. \right. \right. \\
&\quad \left. \left. \left. - \frac{(b_e+1)\sigma_{11}}{2\hat{\alpha}} \right] + \frac{1}{2} [A\sigma_{11} + B\sigma_{21} + C\sigma_{31}] \right\} \right)^{-1}
\end{aligned}$$

The terms in $I(x)$ are same as explained earlier. Therefore, the Bayesian estimator of α under the general entropy loss function is given by

$$\begin{aligned}
\hat{\alpha}_{\text{GELF}} &= [I(x)]^{-\frac{1}{b_e}} \\
&= \left(\hat{\alpha}^{-b_e} \left\{ 1 - \frac{b_e}{\hat{\alpha}} \left[\left(\frac{a_2-1}{\hat{\alpha}} - b_2 \right) \sigma_{11} + \left(\frac{a_3-1}{\hat{\beta}} - b_3 \right) \sigma_{12} + \left(\frac{a_1-1}{\hat{\gamma}} - b_1 \right) \sigma_{13} \right. \right. \right. \\
&\quad \left. \left. \left. - \frac{(b_e+1)\sigma_{11}}{2\hat{\alpha}} \right] + \frac{1}{2} [A\sigma_{11} + B\sigma_{21} + C\sigma_{31}] \right\} \right)^{-\frac{1}{b_e}}
\end{aligned}$$

The general entropy loss function for α is

$$L(\alpha, \hat{\alpha}_{\text{GELF}}) \propto \left(\frac{\hat{\alpha}_{\text{GELF}}}{\alpha} \right)^{b_e} - b_e \ln \left(\frac{\hat{\alpha}_{\text{GELF}}}{\alpha} \right) - 1$$

Bayesian Estimators of α , β , and γ using Non-Informative Prior (Jeffreys' Prior)

Referring to equation (11), the joint posterior distribution for α, β, γ using Jeffreys' prior, i.e. non-informative prior, is given by

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$$g_J^*(\alpha, \beta, \gamma | x) = \frac{\gamma^{n-1} \beta^{n-1} \alpha^{n\beta-1} \prod_{i=1}^n x_i^{-(\beta+1)} \exp\left(-\gamma \alpha^\beta \sum_{i=1}^n x_i^{-\beta}\right)}{\int \int \int_0^\infty \gamma^{n-1} \beta^{n-1} \alpha^{n\beta-1} \prod_{i=1}^n x_i^{-(\beta+1)} \exp\left(-\gamma \alpha^\beta \sum_{i=1}^n x_i^{-\beta}\right) \partial \alpha \partial \beta \partial \gamma}$$

The form of posterior distribution is complex, so the evaluation of the Bayesian estimators of α, β, γ is tedious. Therefore, using Lindley's approximation method, the Bayesian estimators of the parameters are obtained in the case of symmetrical and asymmetrical loss functions for Jeffreys' prior. The expression for Bayesian estimators in this case will also depend upon the various loss functions.

(i) Using Lindley's approximation, Bayesian estimators of α, β, γ under squared error loss function is

$$\begin{aligned}\hat{\alpha}_{\text{SELF}} &= I(x) = E[u(\alpha, \beta, \gamma)] = E(\alpha); u(\alpha, \beta, \gamma) \text{ is a function of } \alpha \\ \hat{\beta}_{\text{SELF}} &= I(x) = E[u(\alpha, \beta, \gamma)] = E(\beta); u(\alpha, \beta, \gamma) \text{ is a function of } \beta \\ \hat{\gamma}_{\text{SELF}} &= I(x) = E[u(\alpha, \beta, \gamma)] = E(\gamma); u(\alpha, \beta, \gamma) \text{ is a function of } \gamma\end{aligned}$$

where

$$\begin{aligned}I(x) &= E[u(\alpha, \beta, \gamma)] \\ &= u(\hat{\alpha}, \hat{\beta}, \hat{\gamma}) + (u_1 a_1 + u_2 a_2 + u_3 a_3 + a_4 + a_5) + \frac{1}{2} [A(u_1 \sigma_{11} + u_2 \sigma_{12} + u_3 \sigma_{13}) \\ &\quad + B(u_1 \sigma_{21} + u_2 \sigma_{22} + u_3 \sigma_{23}) + C(u_1 \sigma_{31} + u_2 \sigma_{32} + u_3 \sigma_{33})]\end{aligned}$$

The expression of ρ depends on prior density, therefore ρ is computed for non-informative prior and given below.

The log of joint prior density (ρ) in the case of non-informative prior, referring to equation (10), is

$$\begin{aligned}\rho &= \log(g_J(\alpha, \beta, \gamma)) = -\log \gamma - \log \alpha - \log \beta \\ \rho_1 &= \frac{\partial \rho}{\partial \alpha} = -\frac{1}{\alpha} \\ \rho_2 &= \frac{\partial \rho}{\partial \beta} = -\frac{1}{\beta}\end{aligned}$$

$$\rho_3 = \frac{\partial \rho}{\partial \gamma} = -\frac{1}{\gamma}$$

(ii) The Bayesian estimators of β and γ under the LINEX loss function are given by

$$\hat{\beta}_{\text{LINEX}} = -\frac{1}{c} \ln[I(x)], I(x) = E(e^{-c\beta})$$

$$\hat{\gamma}_{\text{LINEX}} = -\frac{1}{c} \ln[I(x)], I(x) = E(e^{-c\gamma})$$

(iii) The Bayesian estimator of α under the general entropy loss function is given by

$$\hat{\alpha}_{\text{GELF}} = [I(x)]^{\frac{-1}{b_e}}, I(x) = E(\alpha^{-b_e})$$

The terms in $I(x)$ are the same as already explained.

Table 1. Bayesian estimators of α, β, γ using Jeffreys' prior

Param.	Estimators of α, β, γ under SELF, LINEX and GELF
α	$\hat{\alpha}_{\text{SELF}} = \hat{\alpha} - \frac{\sigma_{11}}{\hat{\alpha}} - \frac{\sigma_{12}}{\hat{\beta}} - \frac{\sigma_{13}}{\hat{\gamma}} + \frac{1}{2} [A\sigma_{11} + B\sigma_{21} + C\sigma_{31}]$ $\hat{\alpha}_{\text{GELF}} = \left(\hat{\alpha}^{-b_e} \left\{ 1 + \frac{b_e}{\hat{\alpha}} \left[\frac{\sigma_{11}}{\hat{\alpha}} + \frac{\sigma_{12}}{\hat{\beta}} + \frac{\sigma_{13}}{\hat{\gamma}} + \frac{(b_e + 1)\sigma_{33}}{2\hat{\alpha}} \right] + \frac{u_{ae1}}{2} [A\sigma_{11} + B\sigma_{21} + C\sigma_{31}] \right\} \right)^{\frac{-1}{b_e}}$
β	$\hat{\beta}_{\text{SELF}} = \hat{\beta} - \frac{\sigma_{21}}{\hat{\alpha}} - \frac{\sigma_{22}}{\hat{\beta}} - \frac{\sigma_{23}}{\hat{\gamma}} + \frac{1}{2} [A\sigma_{12} + B\sigma_{22} + C\sigma_{32}]$ $\hat{\beta}_{\text{LINEX}} = \left(\hat{\beta} - \frac{1}{c} \ln \left\{ 1 + c \left[\frac{\sigma_{21}}{\hat{\alpha}} + \frac{\sigma_{22}}{\hat{\beta}} + \frac{\sigma_{23}}{\hat{\gamma}} + \frac{c}{2} \sigma_{22} - \frac{1}{2} [A\sigma_{12} + B\sigma_{22} + C\sigma_{32}] \right] \right\} \right)$
γ	$\hat{\gamma}_{\text{SELF}} = \hat{\gamma} - \frac{\sigma_{31}}{\hat{\alpha}} - \frac{\sigma_{32}}{\hat{\beta}} - \frac{\sigma_{33}}{\hat{\gamma}} + \frac{1}{2} [A\sigma_{13} + B\sigma_{23} + C\sigma_{33}]$ $\hat{\gamma}_{\text{LINEX}} = \left(\hat{\gamma} - \frac{1}{c} \ln \left\{ 1 + c \left[\frac{\sigma_{31}}{\hat{\alpha}} + \frac{\sigma_{32}}{\hat{\beta}} + \frac{\sigma_{33}}{\hat{\gamma}} + \frac{c}{2} \sigma_{33} - \frac{1}{2} [A\sigma_{13} + B\sigma_{23} + C\sigma_{33}] \right] \right\} \right)$

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By using these expressions of different loss functions, i.e. the squared error loss function, LINEX loss function, and general entropy loss function, Bayesian estimators of α, β, γ are obtained for non-informative prior and presented in Table 1.

Semi-Bayesian Approach

Generalized maximum likelihood estimators are obtained for the parameters of GIWD using informative and non-informative prior. It is known generalized maximum likelihood estimators are maximum likelihood estimators of the posterior distribution.

Generalized Maximum Likelihood Estimators of Parameters (α, β, γ) of Generalized Inverse Weibull Distribution using Informative Prior

Assuming independence among parameters, the posterior distribution using informative prior is given by

$$g_I^*(\alpha, \beta, \gamma | x) \propto \gamma^{n+a_1-1} \beta^{n+a_3-1} \alpha^{n\beta+a_2-1} \prod_{i=1}^n x_i^{-(\beta+1)} \exp\left(\left(-\gamma\alpha^\beta \sum_{i=1}^n x_i^{-\beta}\right) - \gamma b_1 - \alpha b_2 - \beta b_3\right)$$

The log of the posterior distribution is

$$\begin{aligned} \ln(g_I^*(\alpha, \beta, \gamma | x)) &= (n+a_1-1)\ln(\gamma) + (n+a_3-1)\ln(\beta) + (n\beta+a_2-1)\ln(\alpha) \\ &\quad - (\beta+1) \sum_{i=1}^n \ln(x_i) - \gamma\alpha^\beta \sum_{i=1}^n x_i^{-\beta} - b_1\gamma - b_3\beta - b_2\alpha \end{aligned}$$

Taking partial derivatives with respect to α, β , and γ gives likelihood equations

$$\frac{\partial \ln(g_I^*(\alpha, \beta, \gamma | x))}{\partial \alpha} = \frac{n\beta+a_2-1}{\alpha} - \beta \sum_{i=1}^n x_i^{-\beta} - b_2 = 0 \quad (41)$$

$$\begin{aligned}
& \frac{\partial \ln(g_1^*(\alpha, \beta, \gamma | x))}{\partial \beta} \\
&= \frac{n + a_3 - 1}{\beta} + n \ln(\gamma) - \ln\left(\sum_{i=1}^n x_i\right) - \gamma \alpha^\beta \sum_{i=1}^n x_i^{-\beta} \left(\ln \gamma - \ln\left(\sum_{i=1}^n x_i\right)\right) - b_3 \quad (42) \\
&= 0
\end{aligned}$$

$$\frac{\partial \ln(g_1^*(\alpha, \beta, \gamma | x))}{\partial \gamma} = \frac{n + a_1 - 1}{\gamma} - \alpha^\beta \sum_{i=1}^n x_i^{-\beta} - b_1 = 0 \quad (43)$$

By solving these three equations simultaneously, the GMLE of α , β , and γ are obtained. As the expressions are not in explicit form, one may use various numerical methods to obtain the estimators, e.g. Newton Raphson's method (Raphson, 1690) or the Optim function in R (Nash, 1990). In the present study, the latter approach, i.e. Optim function in R, is used to obtain the estimators of the parameters (α, β, γ) of GIWD.

Note: Generalized maximum likelihood estimators of parameters (α, β, γ) of the GIWD using Jeffreys' prior are computed in a similar manner by taking the values of the hyperparameters to be zero ($a_1 = b_1 = a_2 = b_2 = a_3 = b_3 = 0$).

Optimization of Hyperparameters

The hyperparameters, the parameters used in the informative prior, also play an important role in the simulation exercise. So, the choice of hyperparameters is crucial and important to any simulation study. In literature, there are many approaches to estimate these hyperparameters, like using their maximum likelihood estimators, choosing hyperparameters randomly, min/max approach, estimating from past data, or the method of elicitation using a prior predicative distribution (Sinha & Howlander, 1980; Ali, Aslam, Abbas, & Kazmi, 2012; Aslam, 2003). In the previous sections, the min/max approach (Sinha & Howlander, 1980) is used to check the robustness of hyperparameters. In the present section, a scaled total misfit technique is used, which leads to the best fitted hyperparameters out of random choice of some combination of hyperparameters with the parameters of distribution.

A Scaled Total Misfit Measure (Park, 2005)

First define a scaled misfit measure from a quantile estimation method as follows:

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$$A = \sum_q \frac{(\hat{x}_q - x_q)^2}{x_q}$$

where x_q is the true value for the q -quantile and \hat{x}_q is the estimated q -quantile computed from GIWD. This measure can be viewed as the chi-square goodness-of-fit criterion for numeric estimates of quantiles.

Now the above misfit measure is extended to various sample sizes. This measure is computed for N , the number of Monte Carlo samples for different sample sizes n (denoted by N_n), and then averaged:

$$B_n = \sum_{i=1}^{N_n} \frac{A_i}{N_n}$$

Here, A_i is measure computed from the i^{th} Monte Carlo sample.

Define, for a distribution, scaled total misfit measure (STMM) as a function of a given prior and estimation method:

$$Q(\pi, M) = \sum_{\theta} Q_{\theta}(\pi, M)$$

where π is the prior, M is the method of estimation of the parameter θ , and

$$\begin{aligned} Q_{\theta}(\pi, M) &= \sum_n B_n \\ &= \frac{\sum_n \sum_{i=1}^{N_n} \sum_q \frac{(\hat{x}_q - x_q)^2}{x_q}}{N_n} \end{aligned}$$

θ is the parameters of the distribution we are using for quantile estimation. Choose the best prior (π) and M which minimizes $Q(\pi, M)$ among several M 's and classes of priors (π).

Simulation Study

A simulation study is conducted in two parts: firstly, for the selection of the best hyperparameters as per scaled total misfit measure; and secondly to see the relative

efficiency of Bayesian, semi-Bayesian (GMLE's), and non-Bayesian (MLE's) approach. The coding and the analysis are performed using the R programming language.

Random variables from GIWD have been generated using the transformation

$$t = \alpha \left[\frac{-\ln(u)}{\gamma} \right]^{-\frac{1}{\beta}}$$

where u has uniform $U(0, 1)$ distribution.

Table 2. Misfit measure for α

$a_2 = b_2$	$\alpha = 0.5$	1.0	1.5	2.0	2.5	3.0	3.5	4.0	sum Q(π , M)
1	10.48	12.76	8.48	7.67	8.66	8.43	7.80	9.55	73.83
2	9.49	9.36	8.16	7.82	7.92	9.14	9.08	10.24	71.21
3	12.00	9.65	10.61	6.95	7.18	8.70	10.33	9.81	75.23
4	9.78	8.65	7.38	8.84	7.04	10.13	8.17	7.38	67.37
6	9.29	8.82	8.96	9.44	10.59	6.45	7.22	6.82	67.59
8	8.99	8.49	7.91	12.01	8.12	9.23	8.13	8.26	71.14

Table 3. Misfit measure for β

$a_3 = b_3$	$\beta = 0.5$	1.0	1.5	2.0	2.5	3.0	3.5	4.0	sum Q(π , M)
1	64.18	26.44	3.99	2.09	2.27	1.21	0.89	0.72	101.79
2	56.74	21.33	6.60	3.16	2.60	1.44	1.06	0.82	93.75
3	31.36	38.56	7.49	3.65	2.17	1.90	1.26	1.06	87.45
4	40.40	24.98	6.48	2.93	4.44	1.92	1.37	0.86	83.38
6	41.20	37.52	8.72	2.57	1.61	2.05	1.89	1.87	97.43
8	33.47	42.70	9.08	6.09	3.72	3.49	2.01	1.53	102.09

Table 4. Misfit measure for γ

$a_1 = b_1$	$\gamma = 0.5$	1.0	1.5	2.0	2.5	3.0	3.5	4.0	sum Q(π , M)
1	6.99	7.39	10.36	8.43	7.59	6.92	7.46	8.64	63.78
2	7.31	8.57	7.39	7.18	8.70	7.28	6.17	6.57	59.17
3	8.91	8.01	8.30	8.35	8.42	6.79	9.04	7.83	65.65
4	8.93	10.38	8.71	8.38	7.88	10.95	6.47	7.54	69.24
6	8.03	7.76	8.17	8.76	6.02	7.73	5.76	6.58	58.81
8	10.61	9.77	7.61	8.13	8.89	8.03	7.91	6.30	67.25

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The different sample sizes $n = 10, 20, 40, 80, 100, 150$ and quantiles $q = 0.1, 0.2, \dots, 0.9$ are used to compute misfit measure, and $N_n = 1000$ for different combinations of parameters and hyperparameters. The method used in misfit measure is the generalized maximum likelihood estimator and tables for the three parameters are given in Tables 2-4.

From Tables 2-4, the hyperparameter having minimum sum are selected for the further study:

- (i) optimum selection of hyperparameters for α is $a_2 = b_2 = 4$ (from Table 2)
- (ii) optimum selection of hyperparameters for β is $a_3 = b_3 = 4$ (from Table 3)
- (iii) optimum selection of hyperparameters for γ is $a_1 = b_1 = 6$ (from Table 4)

Relative Efficiency of Various Approaches

The mean square errors are computed using maximum likelihood, generalized maximum likelihood, and Bayesian with Jeffreys' and informative prior for the parameters which are obtained using squared error loss function, LINEX loss function, and general entropy loss function. These estimated losses are computed using 10,000 Monte Carlo simulations for different sample sizes $n = 30, 50, 100$ with parameter combinations $\alpha = \beta = \gamma = 2.5, 4$. The combinations of hyperparameters are

- (i) $a_2 = b_2 = 4$ for α ,
- (ii) $a_3 = b_3 = 4$ for β ,
- (iii) $a_1 = b_1 = 6$ for γ

taken for the simulation study according to misfit measure.

Comparison of Non-Bayesian Approach (MLE's) and Semi-Bayesian Approach (GMLE's)

The mean square errors of estimators using non-Bayesian approach (MLE's) and semi-Bayesian approach (GMLE's) are computed for parameters of GIWD and presented in Tables 5-7.

Table 5. Mean square error for α ($\beta = \gamma = 2.5$)

n	α	MLE	GMLE	
			Jeffreys' prior	Informative prior*
30	2.5	0.03709354	0.03585720	0.03593395
	4.0	0.10401380	0.09482988	0.09537666
50	2.5	0.02950446	0.02102547	0.02127104
	4.0	0.05882332	0.05315694	0.05396281
100	2.5	0.01965800	0.01012477	0.01024923
	4.0	0.02736852	0.02606220	0.02612554

Note: * $a_2 = b_2 = 4$ **Table 6.** Mean square error for β ($\alpha = \gamma = 2.5$)

n	α	MLE	GMLE	
			Jeffreys' prior	Informative prior*
30	2.5	0.93413690	0.66481748	0.72403600
	4.0	3.26021101	1.71687667	1.88861660
50	2.5	0.83981996	0.64929604	0.71147050
	4.0	2.70095544	1.68945720	1.82528550
100	2.5	0.76733404	0.63749605	0.69391660
	4.0	2.25106582	1.61900318	1.78274770

Note: * $a_3 = b_3 = 4$ **Table 7.** Mean square error for γ ($\alpha = \beta = 2.5$)

n	α	MLE	GMLE	
			Jeffreys' prior	Informative prior*
30	2.5	0.33456618	0.21560620	0.24933030
	4.0	0.98070166	0.55235300	0.63657320
50	2.5	0.18460151	0.12808220	0.14161840
	4.0	0.90368559	0.32289400	0.35839910
100	2.5	0.07845281	0.06577740	0.06599840
	4.0	0.32742309	0.26368640	0.31544770

Note: * $a_1 = b_1 = 6$

From Tables 5-7, it is observed that

- GMLE's using Jeffreys' prior and informative prior have less mean square error in comparison with MLEs, i.e. semi-Bayesian approaches (GMLEs) give better estimators than non-Bayesian approaches (MLEs).

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- (ii) In the case of the semi-Bayesian approach, GMLEs using Jeffreys' prior perform better in comparison with informative prior as they have smaller mean square error for all the parameters.

Table 8. Mean square error for α ($\beta = \gamma = 2.5$)

n	α	Jeffreys' prior	Informative prior		
			SELF	LINEX	
			$a_3 = b_3 = 4$	$c = 1$	$c = -1$
30	2.5	0.00075143	0.00018920	0.00004106	0.00012588
	4.0	0.03746346	0.01278650	0.00486047	0.00194826
50	2.5	0.00045097	0.00012100	2.398339E-05	7.276694E-05
	4.0	0.00978687	0.00500640	0.00314538	0.00053285
100	2.5	7.648816E-05	0.00004520	4.668325E-06	2.468399E-05
	4.0	0.00385535	0.00075750	9.676239E-05	5.809497E-05

Table 9. Mean square error for β ($\alpha = \gamma = 2.5$)

n	α	Jeffreys' prior	Informative prior		
			SELF	LINEX	
			$a_3 = b_3 = 4$	$c = 1$	$c = -1$
30	2.5	0.01394930	0.00216250	0.00145047	0.00125890
	4.0	1.61635760	1.59734500	0.15938417	0.35485000
50	2.5	0.01186860	0.00061610	1.653527E-04	1.352328E-04
	4.0	1.60876570	1.50787870	0.00193841	0.00128650
100	2.5	4.107887E-03	0.00016530	1.119729E-04	1.092251E-04
	4.0	1.57676760	1.46878800	0.00013869	0.00013300

Table 10. Mean square error for γ ($\alpha = \beta = 2.5$)

n	α	Jeffreys' prior	Informative prior		
			SELF	LINEX	
			$a_3 = b_3 = 4$	$c = 1$	$c = -1$
30	2.5	0.02403470	0.00986600	0.00847382	0.00817383
	4.0	0.52561340	0.46056750	0.09273483	0.85273521
50	2.5	0.01012250	0.00326450	0.00278372	0.00283672
	4.0	0.32068780	0.31650770	0.01873292	0.01737367
100	2.5	0.00492160	0.00100720	0.00097364	0.00098215
	4.0	0.11896230	0.10766540	0.00456722	0.00635622

Full-Bayesian Estimators (Comparison of Symmetrical and Asymmetrical Loss Functions)

Using the above parameter combinations, mean square errors are computed for Bayesian estimators of parameters of GIWD under the assumptions of squared error loss function (symmetrical), LINEX loss function, and general entropy loss functions (asymmetrical) using Jeffreys' and informative prior.

It is seen from the Tables 8-10 that

- (i) Bayesian estimators with informative prior, along with the choice of hyperparameter according to misfit procedure, are found to perform better as compared to Bayesian estimators with Jeffreys' prior because these lead to smaller mean square error for all the estimators for small as well as large sample sizes.
- (ii) In the case of informative prior, asymmetrical loss functions (LINEX, entropy loss function) perform better as compared with the symmetrical loss function (SELF) for all the three parameters.
- (iii) When sample size increases, mean square error decreases in all cases.

Real Data Examples

The data of percentage growth of per capita net state domestic product at current prices 2010-11 is taken from Directorate of Economics & Statistics of respective State Governments, and All-India, CSO as on August 14, 2012. The data fits well to GIWD with p -value 0.637 of the Kolmogorov-Smirnov test at 5% level of significance. The Bayes estimators obtained using semi-Bayesian (GMLE) and full Bayesian approaches have been compared with the classical approach (MLE) to see their relative efficiency. The analysis is done using both informative and non-informative priors and three loss functions, SELF, LINEX loss function, and GELF, along with the same choices of hyperparameters as taken earlier in the case of simulation work. The results obtained are given in Tables 11 and 12.

Table 11. Mean square error of α, β, γ (SELF)

Param.	MLE	GMLE			
		Jeffreys' prior	Informative prior	Jeffreys' prior	Informative prior
α	0.0888790000	0.0804255000	0.8427397000	0.0124469000	0.0058104080
β	1.8778689000	1.8016903500	1.9889200000	0.0358189800	8.78053600E-04
γ	0.0187688000	0.0174003300	0.0437355000	0.0095675800	0.0077583277

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Table 12. Mean square error of α , β , γ (LINEX and GELF)

Loss function	Parameter	Hyperparameter	Informative prior	
			$b_e = 2$	$b_e = -2$
GELF	α	$a_2 = b_2 = 4$	0.01034252	0.00151386

Loss function	Parameter	Hyperparameter	Informative prior	
			$c = 1$	$c = -1$
LINEX	β	$a_3 = b_3 = 4$	0.02345112	0.00043727
	γ	$a_1 = b_1 = 6$	0.00093425	0.00086721

It is seen from Tables 11-12 that

- (i) Bayesian estimators with informative prior perform better as compared to Bayesian estimators with Jeffreys' prior.
- (ii) In the case of informative prior, asymmetrical loss functions (LINEX, entropy loss function) perform better as compared with the symmetrical loss function (SELF) for all three parameters.

As it is obvious the findings from the analysis of real life example are in accordance with those of simulation study.

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